Stability of Zero Production Under Life-Cycle Savings

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INTRODUCTION AND SUMMARY

The no-bequest life-cycle growth model has been the subject of some theoretical interest since Diamond (1965) introduced production into Samuelson's (1958) two-period consumption-loan model and since Meade (1966) incorporated Modigliani's (1966) no-bequest life-cycle model into the Solow (1956) growth model. More recently, the no-bequest life-cycle growth model has served as the basis for the analysis of Social Security by Feldstein (1974) and his associates at the National Bureau of Economic Research. In this note I will establish under certain broad assumptions that the equilibrium of zero production in such models is locally stable.

To establish this result, I will begin by reviewing the stability condition of growth models. That condition will be applied to the two-period model of Diamond and then to the continuous time model investigated by Meade. Finally, the result will be explained by comparison of the no-bequest life-cycle model with models of proportional and classical savings.

1. THE CONDITION FOR STABILITY OF ZERO PRODUCTION

Suppose technology can be approximated by a constant returns CES production function in the vicinity of \( k = 0 \),

\[
f(k) = \gamma \left[ \delta k^{-\rho} + (1 - \delta) \right]^{-1/\rho}
\]

where \( k \) is the capital-labour ratio, \( f(k) \) is output per unit labour, and where labour may be measured in efficiency units to accommodate labour-augmenting technical progress.

If the elasticity of substitution, \( \sigma = 1/1 + \rho \), exceeds unity, output is strictly positive for all \( k \geq 0 \). For \( \sigma \leq 1 \), \( f(0) \) is a point of zero production. This point is a steady-state equilibrium: net savings is impossible since output is zero, and net dissavings is impossible since \( k = 0 \).

The condition for local stability of such an equilibrium is

\[
\lim_{k \to 0} \frac{S}{k} < n
\]

where \( S \) is a measure of the net change in capital, per unit labour, and \( n \) is a measure of the natural rate of growth. With appropriate definitions of \( S \) and \( n \), to be specified below, this condition will serve us for discrete time, finite intervals of continuous time, or infinitesimal intervals of continuous time.

In cases where the set of preferences, expectational assumptions, and of history is sufficient to uniquely determine \( S \) from \( k \), the traditional Solow diagram will illustrate. Zero production is stable if and only if the slope of the savings curve, as it emanates from the origin, is less than \( n \). The proofs presented below do not require such assumptions.
2. THE TWO-PERIOD MODEL

Consider the Diamond two-period no-bequest life-cycle model. Individuals work in the first period of life, but not in the second. In each period, consumption occurs after production. The no-bequest life-cycle assumption is that utility is a function only of present and future consumption. The retired generation, therefore, consumes its capital along with its profit income immediately following production. The younger generation is entirely dependent on wage income, since it is endowed with no physical capital. Savings therefore come entirely out of the younger generation’s wage income, while the retired generation dissaves. We can then write

$$-1 \leq \frac{S}{k} \leq \frac{w - k}{k}$$

where $S$ is the change in capital per period, per unit labour. The lower bound on $S/k$ is set by the extreme case where all output plus the initial capital stock is consumed in the period. The upper bound is set by the opposite extreme case, where individuals wait until retirement before consuming anything.

To evaluate the limit of $S/k$ as $k \to 0$, we assume marginal productivity payments. Then,

$$\lim_{k \to 0} \frac{w(k)}{k} = \lim_{k \to 0} \gamma(1 - \delta)k^{\rho}[\delta + (1 - \delta)k^{\rho}]^{-(1 + \rho)/\rho} = 0$$

for $\sigma < 1$ ($\rho > 0$). Therefore,

$$\lim_{k \to 0} \frac{S}{k} = -1 < n$$

where $n$ is the simple rate of natural growth per period. Therefore, in the vicinity of $k = 0$, the capital-labour ratio falls each period and the system converges monotonically on zero production. The zero production equilibrium is locally stable for any two-period no-bequest life-cycle model with $\sigma < 1$.

**Remark 1.** This result is independent of preferences and also of expectational assumptions.

**Remark 2.** The singular case of $\sigma = 1$ (Cobb-Douglas) is the only CES case where a point of zero production exists but is unstable.

**Remark 3.** For particular sets of preferences and expectational assumptions, one can extend the local analysis to global analysis. For example, under myopic expectations with $\sigma < 1$, if there are any positive equilibria, there will typically be two: one unstable and
one locally stable, in addition to that of zero production. Figure 1(b) above is representative. The economy would converge on $k^{**}$ or on zero production depending on whether initial $k \geq k^*$. Under fixed coefficients ($\sigma = 0$), there may be a single positive equilibrium, which is unstable.\(^1\)

3. THE CONTINUOUS TIME MODEL

In the continuous time model, individuals live $T$ years. Here we define $S$ to be the change in capital over an interval of $T$ years, divided by the initial labour force. The natural growth rate, $n$, is defined correspondingly as the compounded growth of labour over $T$ years, $n = e^{\epsilon T} - 1$, where $g$ is the instantaneous growth rate of labour in efficiency units. That is, if $S/k < n$, the capital-labour ratio will be lower $T$ years from the initial point.

To put an upper bound on $S/k$, we consider the plan most propitious for capital accumulation, subject to the no-bequest constraint that the owners of the initial capital stock will certainly draw it down within $T$ years. The plan most favourable to its replacement and augmentation would be to postpone as long as possible any consumption of that capital stock and of its product. Such consumption could certainly not all be postponed for $T$ years, since that would imply that the initial capital stock is entirely owned by the youngest cohort (not to mention the extreme privation by all cohorts until time $T$). Nonetheless, the analysis of such an extreme case will provide us with an upper bound adequate to our task.

To calculate this upper bound on $S/k$, we first calculate an upper bound on $S_{T-\epsilon}$, the amount of capital accumulated per unit initial labour, up to the instant before the initial stock is consumed with interest. We then calculate a lower bound to the postponed consumption and subtract it off to get an upper bound to $S$.

$$S_{T-\epsilon} = k_0 \exp \left( \int_0^T f(k_\tau) \frac{d\tau}{k_\tau} \right) - k_0 \equiv k_0 \exp [Tf'(0)] - k_0$$ \hspace{1cm} (6)

where $k_0$ is the initial capital-labour ratio. Up until time $T - \epsilon$, the capital stock grows exponentially at a rate equal to the output-capital ratio, which is less than that which would obtain as $k \to 0$, i.e. $f(k_\tau)/k_\tau < \lim_{k \to 0} f(k)/k = f'(0)$, which is infinite for $\sigma < 1$. The capital-labour ratio grows in this interval at a rate equal to $f(k_\tau)/k_\tau - g$, which is positive over the range with which we are concerned, the range where capital deepening is technically feasible. By time $T - \epsilon$, the capital-labour ratio reaches

$$k_{T-\epsilon} = \exp (-gT)[S_{T-\epsilon} + k_0] \equiv k_0 \exp [T(f'(0) - g)].$$ \hspace{1cm} (7)

At time $T$, the initial capital stock, with accrued interest, divided by the initial labour force, equals

$$k_0 \exp \left\{ \int_0^T f'(k_\tau) \frac{d\tau}{k_\tau} \right\} \equiv k_0 \exp \{Tf'(k_{T-\epsilon})\} \equiv k_0 \exp \{Tf'(k_0 \exp [T(f'(0) - g)])\}$$ \hspace{1cm} (8)

by (7) and diminishing marginal productivity. Therefore,

$$-1 \leq \frac{S}{k_0} \leq \exp [Tf'(0)] - 1 - \exp \{Tf'(k_0 \exp [T(f'(0) - g)])\}.$$ \hspace{1cm} (9)

Clearly, then

$$\lim_{k_0 \to 0} \frac{S}{k_0} = -1 < n.$$ \hspace{1cm} (10)

Therefore, in the vicinity of $k = 0$, the capital-labour ratio must reach a new low at least once every $T$ years. Convergence on zero production need not be monotonic, but the amplitude of $k$'s oscillations must also vanish asymptotically. This is shown by (7), for if we
take \( k_0 \) to be an historical low, the maximum that can be reached before another new low is established is \( k_{T-\epsilon} \), which vanishes with \( k_0 \). The zero production equilibrium is locally asymptotically stable for any continuous time no-bequest life-cycle model with \( \sigma < 1 \).

**Remark 4.** This result is independent of preferences, expectational assumptions, and also of the history of \( k \) before it enters the vicinity of zero.

**Remark 5.** Again, the singular case of \( \sigma = 1 \) (Cobb–Douglas) is the only CES case where a point of zero production exists but is unstable.

**Remark 6.** As in the two-period case, if \( \sigma < 1 \), there will typically be two positive equilibria in addition to the locally stable zero equilibrium.

4. CONCLUSION

We have seen that for \( \sigma < 1 \), the equilibrium of zero production is locally stable for any positive \( n \). In other words, for \( \sigma < 1 \), “takeoff” is impossible under life-cycle savings. Some light may be shed on this result by comparing it with a model of classical savings.

Suppose the average propensities to save out of income from labour and capital are given by bounded functions of \( k \), \( s_w(k) \) and \( s_c(k) \). (The Solow model is a special case, where \( s_w(k) = s_c(k) = s \), a constant.) Then

\[
\lim_{k \to 0} \frac{S}{k} = \lim_{k \to 0} \left( s_c(k)f'(k) + \frac{s_w(k)w(k)}{k} \right) = s_c(0)f'(0). \tag{11}
\]

As \( k \to 0 \), wage income per unit capital vanishes, by (4), so all that matters in the vicinity of zero production is the propensity to save out of profits.

The classical view is that this propensity will certainly be positive, so \( s_c(0)f'(0) > n \) for some positive \( n \). That is, if owners of capital are inclined to add to their capital, rather than draw it down, then zero production will be unstable for some positive \( n \).

What distinguishes the no-bequest life-cycle model is that owners of capital, who tend to be retired, draw down their capital. The two-period model of Section 2, above, is illustrative—there \( s_c(0) = -1/f'(0) \). The tendency of owners of capital to liquidate their holdings renders zero production stable.

The significance of the no-bequest assumption should now be apparent. In the presence of bequests, people in various stages of their life-cycle will be owners of capital, and many of them may be inclined to save: younger owners of capital may save for future consumption, while both younger and older owners of capital may save to leave large estates. All of this would tend to make the point of zero production unstable. It would seem that if a private economy is to embark on a growth path which lasts more than one life-cycle, then the time horizons of capital owners may have to extend beyond their own life-cycle, to include bequests.

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NOTES
2. The singularity of the Cobb–Douglas case is relevant to Tobin's (1967) influential paper. Using a multi-period life-cycle model with a refined mortality table and analysis of dependents' consumption, he derived realistic approximations to the observed U.S. capital-output ratio and interest rate. His only stable equilibrium is positive because he chose a Cobb–Douglas production function. More generally, Tobin's results are very sensitive to the choice of production function and utility function. See also Farrell (1970) on this point.

REFERENCES